Resonance statistics in a microwave cavity with a thin antenna

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Abstract

We propose a model for scattering in a flat resonator with a thin antenna. The results are applied to rectangular microwave cavities. We compute the resonance spacing distribution and show that it agrees well with experimental data provided the antenna radius is much smaller than wavelengths of the resonance wavefunctions.

The aim of the present letter is twofold. First of all, we want to present a method to treat scattering in systems consisting of components of different dimensionality. To illustrate how it works, we are going then to apply it to the case of a flat microwave cavity to which a thin antenna is attached. We compute the resonance spacing distribution in this system and show that it reproduces experimental data — within certain range of energies but without introducing any free parameters.

Many recent studies involve an analysis, both theoretical and experimental, of spectral and transport properties of systems with a complicated geometry: let us recall various microwave resonators [1, 2, 3, 4, 5] or conductance fluctuations in quantum dots — see [6, 7, 8, 9] and references therein.

Although physically different such systems share many common properties, in particular, chaotic spectral behavior for certain "resonator" shapes. For the sake of definiteness, we shall speak here only about the electromagnetic case considering the experimental setup of the work mentioned above, *i.e.*, a flat microwave cavity which may be regarded as two–dimensional coupled to one or more antennas that supply power compensating radiative loss.

Already the very first experiments [10] revealed a discrepancy with theoretical predictions: the expected Poissonian eigenvalue–spacing distribution for integrable rectangular resonators was found to be distorted near the origin. This effect was later explained [11] as consequence of the perturbation due to the measuring antenna.

A complementary — and more general — point of view is to regard the resonator with attached antennas as a scattering system and to study its transport properties; this approach is also natural in view of the mentioned analogy with conductance of quantum dots to which quantum wire leads are attached. Solving the scattering problem in the described geometry is not easy and one naturally looks for possible model simplifications. Since the antennas are thin one can try to describe them as lines coupled to the planar resonator in which the field obeys the Laplace equation with Dirichlet boundary conditions.

At a glance the different dimensionality of the configuration space parts adds complications to solution of the corresponding wave equation. However, a standard technique based on self–adjoint extensions [12] allows us to reduce the task to matching solutions in different parts. We shall describe it below restricting ourselves to simplest case with one antenna; a more general situation will be discussed in a forthcoming paper [13].

Let us begin with a halfline attached to a plane; we want to couple the corresponding free Hamiltonians in a self–adjoint way. In physical terms this requirement means the current conservation. It will be ensured by boundary conditions (3) given below which are local, and therefore also applicable when the halfline is replaced by a line segment and/or the plane by any nonempty planar region. The halfline–plus–plane system corresponds to the Hilbert space $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}^2)$, *i.e.*, the field is described by pairs $\phi := \binom{\phi_1}{\Phi_2}$ of square integrable functions on the two parts of the configuration space. For the sake of simplicity, and with the quantum mechanical analogy in mind, we shall speak about wavefunctions.

The system has a rotational symmetry, so the partial–wave decompo-

sition may be employed. In particular, the s–wave component $\Phi_2(r,\varphi) = (2\pi)^{-1/2}\phi_2(r)$ is independent of the azimuthal angle. The Hamiltonian acts on the wavefunction components as the Laplacian,

$$H\phi = \begin{pmatrix} -\phi_1'' \\ -\Delta\Phi_2 \end{pmatrix} . \tag{1}$$

The self–adjointness means a proper matching of the components at the junction. A detailed discussion of this problem involving a full classification of admissible boundary conditions was presented in [12]. They can be expressed in terms of generalized boundary values for wavefunctions in the plane [14]

$$\begin{aligned} L_0(\Phi) &:= \lim_{r\to 0} \frac{\Phi(\vec{x})}{\ln r} , \\ L_1(\Phi) &:= \lim_{r\to 0} \left[\Phi(\vec{x}) - L_0(\Phi)\ln r \right] . \end{aligned} \tag{2}$$

Typical boundary conditions then read

$$\begin{aligned} \phi_1'(0-) &= A\phi_1(0-) + BL_0(\Phi_2) , \\ L_1(\Phi_2) &= C\phi_1(0-) + DL_0(\Phi_2) ; \end{aligned} \tag{3}$$

there are additional lower–dimensional families corresponding to the cases where (3) becomes singular [12]. Let us remark that using the appropriate regularized boundary values, one can couple in a similar way a halfline to $\mathbb{R}^3$, and more generally, two regions whose dimensions do not differ by more than two.

The coefficients A, B, C, D were in the last named paper expressed in terms of elements of the unitary matrix relating the deficiency subspaces. In this way they depend on four real parameters; the number reduces to three if we demand the Hamiltonian to be time–reversal invariant. A drawback of these parametrizations is that they do not show explicitly the ranges for the values of the coefficients.

An alternative approach has been advocated recently in [15]. It is based on expressing the wavefunction in the plane as $\Phi_2 = \alpha G_0(\cdot, 0; \sqrt{\lambda_0}) + u(\cdot)$, where G_0 is the free Green's function for some energy value λ_0 which does not belong to the spectrum of H and u is a function regular at the connection point $\vec{x} = 0$. The boundary conditions then tie α and $u(0)$ with

$\phi_1(0-)$, $\phi_1'(0-)$. In general, one can obtain in this way the same set of admissible Hamiltonians as above, however, in the mentioned paper a particular subset corresponding to $A=0$ was discussed and used.

A more straightforward way is to compute the boundary form to H_0^*, the adjoint to H_0 obtained by restriction of a "decoupled" operator, say, the one with $B=C=0$, to functions which vanish at the vicinity of the junction [12]. Since the action of H_0^* is given by the same differential expression (1) we find after a simple integration by parts

$$\begin{aligned}(\phi, H_0^*\psi) - (H_0^*\phi, \psi) \;=&\; \bar\phi_1'(0)\psi_1(0) - \bar\phi_1(0)\psi_1'(0) \\ &+ \lim_{\varepsilon\to 0+} \varepsilon \left(\bar\phi_2(\varepsilon)\psi_1'(\varepsilon) - \bar\phi_2'(\varepsilon)\psi_2(\varepsilon)\right) . \end{aligned} \tag{4}$$

We know from [12] that only the s–wave component in the plane can be coupled nontrivially, hence we may suppose that $\Phi_2(\vec x)$ is independent of the azimuthal angle and

$$\phi_2(\varepsilon) \;=\; \sqrt{2\pi}\,\left[L_0(\Phi_2)\ln\varepsilon + L_1(\Phi_2) + \mathcal{O}(\varepsilon)\,\right] .$$

Using this asymptotic behavior we can rewrite the the last term on the *rhs* of (4) as

$$2\pi\left[L_1(\Phi_2)L_0(\Psi_2) - L_0(\Phi_2)L_1(\Psi_2)\right] ;$$

it is then clear that the boundary form is zero under the conditions (3) with

$$A,\, D \in I\!\!R \qquad \text{and} \qquad B = 2\pi\bar C\,. \tag{5}$$

If the Hamiltonian should be time–reversal invariant, the boundary conditions must not change when passing to complex conjugated wavefunctions; this happens if the off-diagonal coefficients B, C are real.

To proceed further we have specify the coefficients in (3). The choice needs a physical motivation. Since we are interested in transport through such junctions, we shall use a comparison of low–energy scattering properties with those of a more realistic antenna. The reflection and transmission amplitudes for a plane wave e^{ikx} approaching the junction on the halfline were computed in [12]: we have

$$r(k) \;=\; -\,\frac{\mathcal{D}_-}{\mathcal{D}_+}\,, \qquad t(k) \;=\; \frac{2iCk}{\mathcal{D}_+}\,, \tag{6}$$

where

$$\mathcal{D}_\pm \;:=\; (A \pm ik)\left[1+\frac{2i}{\pi}\left(\gamma-D+\ln\frac{k}{2}\right)\right]+\frac{2i}{\pi}\,BC$$

and $\gamma = 0.5772\dots$ is the Euler's constant.

Consider now a semiinfinite cylindrical tube of radius a attached to the plane; we suppose the latter has the central circular area of radius a removed so that the whole surface is topologically equivalent to an infinite cylinder. The scattering on such a surface is certainly affected by the sharp edge at the interface of the two parts but we disregard this fact.

We have a rotational symmetry again, so each partial wave may be treated separately. Suppose that the "longitudinal" component of the incident wavefunction is a plane wave of momentum k. For an orbital quantum number ℓ one has to match smoothly the corresponding solutions

$$f(x) \;:=\; \left\{ \begin{array}{lcl} e^{ikx}+r_a e^{-ikx} & \dots & x \le 0 \\ \\ \sqrt{\frac{\pi kr}{2}}\, t_a H_\ell^{(1)}(kr) & \dots & r \ge a \end{array} \right.$$

We abuse here notation on the *lhs* and employ x as a common denomination of the longitudinal variable on the cylinder and the radial variable in the plane. The matching conditions are solved easily giving

$$r_a(k) \;=\; -\,\frac{\mathcal{D}_-^a}{\mathcal{D}_+^a}\,, \qquad t_a(k) \;=\; 4i\sqrt{\frac{2ka}{\pi}}\,\left(\mathcal{D}_+^a\right)^{-1} \tag{7}$$

with

$$\mathcal{D}_\pm^a \;:=\; (1 \pm 2ika)H_\ell^{(1)}(ka) + 2ka\left(H_\ell^{(1)}\right)'(ka)\;;$$

the Wronskian relation $W(J_\nu(z), Y_\nu(z)) = 2/\pi z$ implies that the unitarity requirement, $|r_a(k)|^2 + |t_a(k)|^2 = 1$, is satisfied.

It follows from the asymptotic properties of Bessel functions [16] that $|t_a(k)|^2 \sim (ka)^{2\ell-1}$ holds for $\ell \ge 1$, so the transmission probability vanishes fast as $k \to 0$ for higher partial waves. On the other hand,

$$H_0^{(1)}(z) \;=\; 1+\frac{2i}{\pi}\left(\gamma+\ln\frac{ka}{2}\right)+\mathcal{O}(z^2\ln z)\,.$$

Substituting into (7) we find that these amplitudes have for $ka \ll 1$ the same leading behavior as (6) provided we put

$$A := \frac{1}{2a}\,, \qquad D := -\ln a\,, \qquad BC := \frac{1}{a}\,; \tag{8}$$

owing to (5) the last relation is in the case of "real" boundary conditions satisfied if

$$B \,=\, 2\pi C \,=\, \sqrt{\frac{2\pi}{a}}\,. \tag{9}$$

In this way we have found values of the coefficients for which the coupling (3) is able to model the scattering behavior of a real antenna as long as the wavelengths involved are much larger than the antenna radius. A similar effect has been noticed recently for point interactions in dimension two or three [17]: long enough waves feel only the size of an obstacle or a drain. It is also worth noting that the coefficient A is always nonzero by (8), so the coupling used in Ref.[15] does not belong to this class.

Now we are ready to describe our model. We consider a resonator in the form of a planar region M to which an antenna is attached at a point $x_0 \in M$; we suppose that at the junctions the wavefunctions are coupled by boundary conditions (3) with the parameter values (8), (9), and that the resonator has hard walls which we model by the Dirichlet condition, $u(\vec{x}) = 0$ at the boundary of M.

We have to match the plane wave combination $e^{ikx} + r\,e^{-ikx}$ on the antenna halfline $\mathbb{R}_-$ with the internal solution $u(\vec{x}) := bG(\vec{x},\vec{x}_1;k)$, where $G(\cdot,\cdot;k)$ is the Green's function of the Dirichlet Laplacian; we assume, of course, that k^2 equals to no eigenvalue of this operator. To make use of (3) we have to know the generalized boundary values $L_i := L_i(u;\vec{x}_0)$; they are

$$L_0 = -\frac{b}{2\pi}\,, \qquad L_1 = b\xi(\vec{x}_0;k)\,, \tag{10}$$

where

$$\xi(\vec{x}_0;k) := \lim_{\vec{x}\to\vec{x}_0}\left[\,G(\vec{x},\vec{x}_0;k) + \frac{\ln|\vec{x}-\vec{x}_0|}{2\pi}\,\right]. \tag{11}$$

Matching then the solutions we find

$$r(k) \,=\, -\,\frac{\pi Z(k)(1-2ika)-1}{\pi Z(k)(1+2ika)-1} \tag{12}$$

with

$$Z(k) \;:=\; \xi(\vec{x}_0;k) - \frac{\ln a}{2\pi} \tag{13}$$

In order to make use of these formulae we have to know the last named quantity. If the region M is compact, the corresponding Dirichlet Laplacian has a purely discrete spectrum. We denote by λ_n amd ϕ_n , respectively, the corresponding eigenvalues and eigenfunctions; without loss of generality the latter can be chosen real. This allows us to express the Green's function,

$$G(\vec{x}_1,\vec{x}_2;k) \;=\; \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}_1)\phi_n(\vec{x}_2)}{\lambda_n - k^2}\,. \tag{14}$$

It diverges, of course, as $\vec{x}_1 \to \vec{x}_2$; we need to know the regularized value (11). A semiclassical argument [18, Sec.XIII.15] makes it possible to asses the rate of the divergence: we have $\lambda_n \approx 4\pi n|M|^{-1}$ and $\langle|\phi_n(\vec{x})|^2\rangle \approx |M|^{-1}$ as $n \to \infty$, where $|M|$ is the area of the resonator. This inspires us to employ the identity

$$\begin{aligned} G(\vec{x}_0+\sqrt{\varepsilon}\vec{n},\vec{x}_0;k) + \frac{\ln\sqrt{\varepsilon}}{2\pi} \;&=\; \\ \sum_{n=1}^{\infty} \left\{ \frac{\phi_n(\vec{x}_0+\sqrt{\varepsilon}\vec{n})\phi_n(\vec{x}_0)}{\lambda_n - k^2} \right. &- \left. \frac{(1-\varepsilon)^n}{4\pi n} \right\}\,, \end{aligned}$$

where $\vec{n}$ is an arbitrary unit vector. The two series diverge as $\varepsilon \to 0$ but their difference remains finite: we have

$$\xi(\vec{x}_0;k) \;=\; \sum_{n=1}^{\infty} \left\{ \frac{|\phi_n(\vec{x}_0)|^2}{\lambda_n - k^2} - \frac{1}{4\pi n} \right\}\,. \tag{15}$$

Let us stress that the exchange of the limit and summation should be taken with a caution, since the series is not uniformly convergent. Replacing, *e.g.*, $\sqrt{\varepsilon}$ by $c\sqrt{\varepsilon}$ we get an additive factor. However, the correct $\xi(\vec{x}_0;k)$ should be independent of the boundary at large negative energies,

$$\lim_{\kappa\to\infty} \left[\xi(\vec{x}_0;i\kappa) - \frac{1}{2\pi}\left(\ln\frac{\kappa}{2} + \gamma\right)\right] \;=\; 0\,,$$

by Ref.[12]. For a specific M as the one considered below, it is straightforward to check that this is true for (15).

We are especially interested in the situation where M is a rectangle $[0,c_1]\times[0,c_2]$, for which we have

$$\phi_{nm}(x,y) = \frac{2}{\sqrt{c_1c_2}}\sin(n\frac{\pi}{c_1}x)\sin(m\frac{\pi}{c_2}y)\,, \tag{16}$$

$$\lambda_{nm} = \frac{n^2\pi^2}{c_1^2}+\frac{m^2\pi^2}{c_2^2}\,; \tag{17}$$

the formulae (12) and (15)–(17) yield a complete solution of the scattering problem in this setting.

In particular, we want to find the resonance spacing distribution which can be compared with the result found experimentally in [11]; we refer to this paper for a more detailed description of the experimental arrangement. By (12) the resonances are given by complex zeros of the denominator, *i.e.*, by solutions of the algebraic equation

$$\xi(\vec{x}_0,k) = \frac{\ln(a)}{2\pi} + \frac{1}{\pi(1+ika)} \tag{18}$$

where a is the radius of the antenna opening. We have solved this equation numerically and evaluated the resonance spacing distribution. In order to mimick the experimental setting we have evaluated the resonance spaing distribution for several rectangular resonators with c_1 and c_2 ranging from 20 to 50 cm. The coupling point $\vec{x}_0$ has been chosen randomly for each billiard and the diameter of the antena was equal to 1 mm. The obtained data has been combined in order to enlarge the statistics. Since the present model was derived under the assumption $ka \ll 1$ we used only the data corresponding to frequencies below 10 GHz for which $ka < 0.1$. In addition, it is known that about 7% of resonances are overlooked in the experiment; to take the missing levels into account we have deleted the same amount of randomly chosen resonances from the family under consideration.

The result is plotted on the Figure 1. The agreement between the model calculations and the experiment data is convincing. We recall that our argument involves no free parameters because the "coupling strength" between resonator and the antenna is fully determined by the size of the latter. On the other hand, the long–wave condition relative to the antenna radius, $ka \ll 1$, is important: the agreement worsens if resonances above 10 GHz are taken into account.

In conclusion, we have described a method to treat the scattering problem in systems with configuration space composed of parts of a different dimensionality. It has been applied to a flat microwave resonator with a thin antenna; the calculated resonance spacing distribution agrees in the long–wave regime with the experimental data.

Acknowledgments

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Figure captions

Figure 1. The numerically evaluated resonance spacing distribution for the rectangular resonator (full line) in comparison with the experimental data taken from [11] (bins). In the inset we sketch the geometric arrangement of the experiment.

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Figure 1